REGULAR ORBITAL MEASURES ON LIE ALGEBRAS

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ABSTRACT. Let H_0 be a regular element of an irreducible Lie Algebra $\mathfrak{g}$, and let μ_{H_0} be the orbital measure supported on O_{H_0}. We show that $\hat{\mu}_{H_0}^k \in L^2(\mathfrak{g})$ if and only if $k > \dim \mathfrak{g}/(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})$.

1. INTRODUCTION

Let G be a compact, connected, simple Lie group and $\mathfrak{g}$ its Lie algebra. It is well known that the non-trivial adjoint orbits in $\mathfrak{g}$ are compact submanifolds of proper dimension, but geometric properties ensure that they generate $\mathfrak{g}$. Consequently, if $H_0 \neq 0$ is in the torus $\mathfrak{t}$ of $\mathfrak{g}$, and μ_{H_0} is the orbital measure supported on the orbit O_{H_0} containing H_0, ie, μ_{H_0} is the unique (up to normalization) G-invariant measure on O_{H_0}, then some convolution power of μ_{H_0} is absolutely continuous to Lebesgue measure on $\mathfrak{g}$ and even belongs to $L^{1+\varepsilon}$ for some $\varepsilon > 0$ (see [8]). In [6], Ragozin showed that dimension of $\mathfrak{g}$ convolution powers sufficed, and this was improved in a series of papers culminating in [2] with the minimal number of convolution powers being $k_G = \operatorname{rank} G$ for the classical simple Lie algebras of type B_n, C_n and D_n and $k_G = \operatorname{rank} G + 1$ for type A_n. There it was also shown that if μ_h was the orbital measure supported on the conjugacy class in G containing the non-central element h, then $\mu_h^{k_G} \in L^2(G)$.

In the simplest case $G = SU(2)$, $\mathfrak{g} = \mathbb{R}^3$ and the adjoint orbits are (the two dimensional) spheres centred at the origin. The sum of

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two such spheres contains an open set and consequently the convolution of any two orbital measures is absolutely continuous [7]. In general, the generic orbits (the so-called regular orbits defined below) have codimension $\operatorname{rank} G$ and two convolution powers of such an orbital measure is absolutely continuous (in either the group or algebra case). Furthermore, for the generic orbital measure μ_h on the group, one can use the Weyl character formula to see that $\mu_h^{k_2} \in L^2(G)$ for $k_2 = 1 + \operatorname{rank} G/(\dim G - \operatorname{rank} G)$ (see [4]) and this fact can be transferred to the Lie algebra setting as well [3].

In this note we give a direct proof that if μ_{H_0} is any generic orbital measure on $\mathfrak{g}$, then $\hat{\mu}_{H_0}^k \in L^2(\mathfrak{g})$ if and only if $k > 1 + \operatorname{rank} \mathfrak{g}/(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})$. The novelty of our approach is our geometric method, involving the root systems, of handling the singularities which arise in the integral of the Fourier transform of the measure.

Products of generic orbital measures are also studied in [1] and [9]; our approach recovers some of what was proven in [9].

2. Definitions and Lemmas

Let T be a maximal torus of G and t be the corresponding subalgebra of $\mathfrak{g}$, also called the torus. Let Φ be the root system of $\mathfrak{g}$ with Weyl group $\mathcal{W}$ and positive roots Φ^+. Choose a base $\Delta = \{\beta_1, \ldots, \beta_n\}$ for Φ and let $\mathfrak{t}^+$ be the associated fundamental Weyl chamber.

$$\mathfrak{t}^+ = \{H \in t : (H, \beta_j) > 0 \text{ for } j = 1, \ldots n\}$$

Given $H_0 \in t$, the adjoint orbit of H_0 is given by

$$O_{H_0} = \{Ad(g)H_0 : g \in G\} \subseteq \mathfrak{g}.$$

If $H_0 \in \mathfrak{t}^+$, then H_0 is called regular and O_{H_0} is a called a regular orbit.

The regular orbital measure, μ_{H_0}, is the G-invariant measure supported on the regular orbit O_{H_0}, normalized so the Harish-Chandra formula gives

$$\widehat{\mu}_{H_0}(H) = \frac{A_{H_0}(H)}{\prod_{\alpha \in \Phi^+}(\alpha, H)} \text{ for } H \in \mathfrak{t}^+$$

where

$$A_{H_0}(H) = \sum_{\sigma \in \mathcal{W}} \operatorname{sgn}(\sigma) e^{i(\sigma(H), H_0)}.$$

As μ_{H_0} is G-invariant, the Weyl integration formula implies that $\hat{\mu}_{H_0}^k \in L^2(\mathfrak{g})$ if and only if

$$\int_{\mathfrak{t}^+} \frac{|A_{H_0}(H)|^{2k}}{\left|\prod_{\alpha \in \Phi^+}(\alpha, H)\right|^{2k-2}} dH < \infty.$$

In this integral some of the inner products (α, H) represent removable singularities on some walls of the Weyl chamber. This is the primary obstacle in studying this integral and we are able to deal with these singularities using geometry and an induction argument.

Specifically, we will relate the integrand near a collection of walls to the integrand for a subroot system. The power of our induction is hidden in the fact that the the integrand is continuous, and so is bounded on any neighborhood of the origin. Several technical problems arise; in fact they are necessary adaptations to the proof of a weaker result (Cor. 1), where the technical results are not necessary. The case of a Lie algebra of type A_2 is surprisingly representative, and the geometric motivation for the results presented here come exclusively from this case.

The notation will get slightly tedious, so we list it all here in advance. Note that from now on we assume Φ is irreducible, but we will consider reducible subroot systems of Φ that are "simple"; these are simply those subroot systems for which a subset of Δ can be chosen as a base.

$\mathfrak{g}$	An irreducible Lie algebra.
Φ	The root system of $\mathfrak{g}$.
$\Delta = \{\beta_1, \ldots, \beta_n\}$	The simple roots of Φ.
Φ^+	The positive roots of Φ.
n	The rank of $\mathfrak{g}$.
$\mathcal{W}$	The Weyl group of Φ.
$\mathfrak{t}^+ = \{H \in \mathfrak{g} : (H, \beta_i) > 0 \text{ for all } i\}$	The fund. Weyl chamber of $\mathfrak{g}$.
Ψ	A simple subroot system of Φ.
$\mathcal{V}$	The Weyl group of Ψ.
$\{\gamma_1, \gamma_2, \ldots, \gamma_m\} \subset \Delta$	A base for Ψ.
Ψ^+	The positive roots of Ψ.
m	Number of simple roots in Ψ.
$\mathfrak{s}^+ = \{H \in \operatorname{span}\Psi : (H, \gamma_i) > 0 \text{ for } i > 1\}$	The fund. Weyl chamber of Ψ.

Recall that every $H \in \mathfrak{s}^+$ can be written as a non-negative linear combination of the simple roots γ_i. This follows from the fact that, in the irreducible case, all entries of the inverse of the Cartan matrix are positive numbers. (See [5], section 13.4, exercise 8.)

We will need to break $\mathfrak{s}^+$ up into the regions

$$R_i = \{H \in \mathfrak{s}^+ : \|H\| \geq 1, (\gamma_i, H) \geq (\gamma_j, H) \text{ for all } j\}.$$

So $\mathfrak{s}^+ \setminus B_1 = \cup_{i=1}^m R_i$. Now if

$$\Psi_1 = \operatorname{span}_{\mathbb{Z}}\{\gamma_2, \ldots, \gamma_m\} \cap \Psi$$

then the roots of Ψ_1 will correspond to removable singularities on the walls of $\operatorname{cl}(R_1)$ when we calculate the above integral with root system Ψ. Now let $\mathcal{V}_1$ be the Weyl group of Ψ_1, and

$$\mathfrak{c}^+ = \{H \in \operatorname{span}\Psi_1 : (H, \gamma_i) > 0 \text{ for } i = 2 \ldots m\}$$

be the fundamental Weyl chamber of Ψ_1. Finally, we define

$$P : \operatorname{span}\Psi \to \operatorname{span}\Psi : H \mapsto \frac{1}{|\mathcal{V}_1|} \sum_{\sigma \in \mathcal{V}_1} \sigma(H).$$

Lemma (1). *Let P be as above. Then*

(i) *$\sigma(P(H)) = P(H)$ for all $\sigma \in \mathcal{V}_1$.*
(ii) *P is the projection from $\operatorname{span}\Psi$ onto $(\operatorname{span}\Psi_1)^{\perp}$. So $I - P$ is the projection from $\operatorname{span}\Psi$ onto $\operatorname{span}\Psi_1$.*
(iii) *$I - P$ in fact maps $\mathfrak{s}^+$ to $\mathfrak{c}^+$.*

(iv) *There are constants $a, b > 0$ so that $\|P(H)\| \geq a\|H\|$ and $\|(I - P)H\| \leq b\|P(H)\|$ if $H \in R_1$.*

Before reading the proof of this result, the reader is encouraged to graphically verify part (ii) for the case $\Phi = A_2$.

Proof. (i) If $\sigma_1 \in \mathcal{V}_1$ then

$$\sigma_1(P(H)) = \frac{1}{|\mathcal{V}_1|} \sum_{\sigma \in \mathcal{V}_1} \sigma_1(\sigma(H)) = \frac{1}{|\mathcal{V}_1|} \sum_{\sigma \in \sigma_1 \mathcal{V}_1} \sigma(H) = P(H).$$

(ii) Write $H = s + r$, where $s \in \operatorname{span} \Psi_1$ and $r \in (\operatorname{span} \Psi_1)^\perp$. If $\alpha \in \Psi_1$ then

$$\sigma_\alpha(r) = r - \frac{2\,(r, \alpha)}{(\alpha, \alpha)} \alpha = r.$$

Since $\mathcal{V}_1$ is generated by reflections of the form σ_α, $\alpha \in \Psi_1$, it follows that $\sigma(r) = r$ for all $\sigma \in \mathcal{V}_1$. Hence

$$P(H) = P(r) + P(s) = r + P(s).$$

If $\alpha \in \Psi_1$ then by (i) $\sigma_\alpha(P(s)) = P(s)$. Since we also have

$$\sigma_\alpha(P(s)) = P(s) - \frac{2\,(P(s), \alpha)}{(\alpha, \alpha)} \alpha$$

we get that $P(s) \in (\operatorname{span} \Psi_1)^\perp$. But $P(s) \in \operatorname{span} \Psi_1$ so $P(s) = 0$. Putting all of this together, we get that $P(H) = r$ is the projection of H onto $(\operatorname{span} \Psi_1)^\perp$.

Of course it follows that $H - P(H)$ is the projection of H onto $\operatorname{span} \Psi_1$.

(iii) If $k > 1$ and $H \in \mathfrak{s}^+$ then

$$(\gamma_k, H - P(H)) = (\gamma_k, H) > 0$$

since $P(H) \in \operatorname{span}\{\gamma_2, \ldots, \gamma_m\}^\perp$.

(iv) Suppose, in order to obtain a contradiction, that $H \in \operatorname{cl}(R_1)$ and $P(H) = 0$. Then $H \in \operatorname{span} \Psi_1$ and $H \in \operatorname{cl}(\mathfrak{s}^+)$ so we can write

$$H = c_2 \gamma_2 + \ldots + c_m \gamma_m$$

with all $c_i \geq 0$. Thus

$$(H, \gamma_1) = c_2\,(\gamma_2, \gamma_1) + \ldots + c_m\,(\gamma_m, \gamma_1).$$

We also have $(\gamma_i, \gamma_j) \leq 0$ if $i \neq j$, so in fact $(H, \gamma_1) \leq 0$. Since $H \in \mathfrak{s}^+$ $(H, \gamma_1) \geq 0$. Combining these we get $(H, \gamma_1) = 0$. From the definition of R_1 we get, for each $i = 1, \ldots, m$, that

$$0 \leq (H, \gamma_i) \leq (H, \gamma_1) = 0$$

which contradicts the fact that $\|H\| \geq 1$. Thus $P(H) \neq 0$ on $\mathrm{cl}(R_1)$. In particular, $P(H)$ is nonzero on the compact set $\mathrm{cl}(R_1) \cap \{H : \|H\| = 1\}$, so there is an $a > 0$ such that $a \leq \|P(H)\|$ if $\|H\| = 1$, $H \in R_1$. Thus we see that $a\|H\| \leq \|P(H)\|$ on R_1. Finally, we can take $b = \frac{1}{a} + 1$. $\square$

We commented earlier that the roots of Ψ_1 will cause problems in R_1 when integrating. As it turns out, all the other roots of Ψ are very well behaved on R_1. (It is quite helpful to think of the roots of Ψ_1 as the "good" roots on R_1, and the roots of $\Psi \setminus \Psi_1$ as the "bad" roots.)

Lemma (2). *There exists $C > 0$ such that for all $\alpha \in \Psi^+ \setminus \Psi_1^+$ and for all $H \in R_1$*

$$(H, \alpha) \geq C\|H\|.$$

Proof. Take $\alpha \in \Psi^+ \setminus \Psi_1^+$. Write $\alpha = \sum a_i \gamma_i$ with all $a_i \geq 0$. Since $\alpha \notin \Psi_1$, $a_1 > 0$. Now if $H \in \mathrm{cl}(R_1)$

$$(H, \alpha) = \sum a_i (H, \gamma_i) \geq a_1 (H, \gamma_1) > 0.$$

Thus the function

$$f(H) = (H, \alpha)$$

is non zero on the compact set $\mathrm{cl}(R_1) \cap \{H : \|H\| = 1\}$. Hence it attains a positive minimum M_α. We can take $C = \min_{\alpha \in \Psi^+ \setminus \Psi_1^+} M_\alpha$. $\square$

We will be interested in subroot systems of Φ of the form

$$\{a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m : a_i \in \mathbb{Z} \text{ for all } i\} \cap \Phi$$

where $\{\alpha_1, \alpha_2, \ldots \alpha_m\} \subset \Delta$. We will call these **simple** subroot systems. Note that Ψ_1 is a simple subroot system of Φ. Simple subroot systems are the only type of subroot systems that will come up in our induction. Restricting our attention to simple subroot systems makes the verification of the following technical lemma easier.

Lemma (3). *Suppose Φ is an irreducible root system, with simple sub-root system Ψ with m simple roots, where $m < n$. Then*

$$\frac{n}{|\Phi|} < \frac{m}{|\Psi|}$$

Proof. When we look at this result for a particular m, it is clearly sufficient to prove it for the largest Ψ with m simple roots. We list these subroot systems in Appendix A, along with the ratios in question. See [5] for basic facts needed about subroot systems. $\square$

It is worth noting that this lemma is not true if we allow Φ to be reducible. For example, consider a subroot system of Lie type B_3 ($\frac{m}{|\Psi|} = \frac{1}{6}$) in a root system of Lie type $B_3 \times A_1 \times A_1 \times A_1$ ($\frac{n}{|\Phi|} = \frac{1}{4}$).

We now set $\epsilon_0 > 0$ to be any number with $\epsilon_0 < \frac{m}{|\Psi|} - \frac{n}{|\Phi|}$ for all proper simple subroot systems Ψ of Φ. We will need this ϵ_0 later for technical reasons.

3. THE MAIN RESULT

Theorem. *Let Φ be an irreducible root system. Then $\hat{\mu}_{H_0}^k \in L^2(\mathfrak{g})$ if and only if $k > 1 + \frac{n}{|\Phi|} = \dim \mathfrak{g}/(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})$.*

Corollary (1). *If μ is a regular orbital measure, then $\hat{\mu}^{\frac{3}{2}} \in L^2(\mathfrak{g})$.*

Corollary (2). *If μ is a regular orbital measure then $\mu^2 \in L^p(\mathfrak{g})$ for all $p < \frac{\dim(\mathfrak{g})}{\operatorname{rank}(\mathfrak{g})}$.*

Proof. Our arguments show that $\hat{\mu}^2 \in L^{p'}$ for $p' < 1 + \frac{n}{|\Phi|}$. By the Hausdorff-Young inequality, $\mu^2 \in L^p$ for all $p < \frac{\dim(\mathfrak{g})}{\operatorname{rank}(\mathfrak{g})}$. $\square$

It is worth noting that Corollary 2 is sharp when $\mathfrak{g} = \mathfrak{su}(2)$ by a result of Ragozin (see [7], Prop A.5).

Proof. (Of main theorem.) We prove a related result for all simple subroot systems Ψ of Φ. All the notation will be as before, including the definition of ϵ_0.

Our induction hypothesis: For all simple proper subroot systems Ψ of Φ, if $k < 1 + \frac{n}{|\Phi|} + \epsilon_0$ then

$$\int_{\mathfrak{s}^+ \cap B_r} \frac{|A_{H_0}(H)|^{2k} dH}{|\prod_{\alpha \in \Psi^+}(\alpha, H)|^{2k-2}} = O(r^{n-(k-1)|\Psi|}).$$

By this we mean that this integral is bounded above, as a function of r, by $Cr^{n-(k-1)|\Psi|}$ for some $C > 0$.

For $m = 1$, Ψ is of Lie type A_1 and we get

$$\int_1^r \frac{|e^{itH_0} - e^{-itH_0}|^{2k}}{|t|^{2k-2}} dt.$$

Hence when $k < 1 + \frac{1}{2}$ the integrand is $O(r^{2-2k})$ and $2 - 2k > -1$. So if $k < 1 + \frac{1}{2}$ the integral is $O(r^{1-2(k-1)})$. Lemma 3 tells us that $1 + \frac{1}{2} = 1 + \frac{n}{|\Phi|} + (\frac{1}{2} - \frac{n}{|\Phi|}) > 1 + \frac{n}{|\Phi|} + \epsilon_0$.

Now we assume the result for all simple subroot systems of rank $m - 1$.

Consider a subroot system Ψ of rank m. We will describe the growth of the integral on R_1. Since we have not specified any particular order among the R_i, and the integrand is continuous, this is sufficient.

Let $\sigma_1, \ldots, \sigma_t$ be representatives from the left cosets of $\mathcal{V}_1 \leq \mathcal{V}$. We break up $|A_{H_0}|$ by cosets of Ψ_1.

$$\int_{R_1 \cap B_r} \frac{|\sum_{j=1}^t \sum_{\sigma \in \mathcal{V}_1} \mathrm{sgn}(\sigma_j \sigma) e^{i(\sigma_j \sigma(H), H_0)}|^{2k} dH}{|\prod_{\alpha \in \Psi^+} (\alpha, H)|^{2k-2}}$$

$$\leq 2^{2k} \sum_{j=1}^t \int_{R_1 \cap B_r} \frac{|\sum_{\sigma \in \mathcal{V}_1} \mathrm{sgn}(\sigma_j \sigma) e^{i(\sigma_j \sigma(H), H_0)}|^{2k} dH}{|\prod_{\alpha \in \Psi^+} (\alpha, H)|^{2k-2}}$$

For convenience we forget about the constant, and just write the term of the σ_j coset. We start by factoring out $\left| \mathrm{sgn}(\sigma_j) e^{i(\sigma_j(P(H)), H_0)} \right| = 1$ to get

$$\int_{R_1 \cap B_r} \frac{|\sum_{\sigma \in \mathcal{V}_1} \mathrm{sgn}(\sigma) e^{i(\sigma_j \sigma(H), H_0) - i(\sigma_j(P(H)), H_0)}|^{2k} dH}{|\prod_{\alpha \in \Psi^+} (\alpha, H)|^{2k-2}}.$$

Since $P(H) = \sigma(P(H))$ (for $\sigma \in \mathcal{V}_1$) and $(\sigma(v), w) = (v, \sigma(w))$ this integral equals

$$\int_{R_1 \cap B_r} \frac{|\sum_{\sigma \in \mathcal{V}_1} \mathrm{sgn}(\sigma) e^{i(\sigma(H - P(H)), \sigma_j(H_0))}|^{2k} dH}{|\prod_{\alpha \in \Psi^+ \setminus \Psi_1^+} (\alpha, H)|^{2k-2} |\prod_{\alpha \in \Psi_1^+} (\alpha, H)|^{2k-2}}.$$

Now we apply Lemma 2 to get the upper bound

$$\int_{R_1 \cap B_r} \frac{C}{\|H\|^{(k-1)(|\Psi| - |\Psi_1|)}} \frac{|\sum_{\sigma \in \mathcal{V}_1} \mathrm{sgn}(\sigma) e^{i(\sigma(H - P(H)), \sigma_j(H_0))}|^{2k} dH}{|\prod_{\alpha \in \Psi_1^+} (\alpha, H)|^{2k-2}}.$$

At this point we can safely replace $\sigma_j(H_0)$ with $H_0' = (I - P)\sigma_j(H_0)$. Since $P(H)$ is orthogonal to Ψ_1, we can change the inner products from (α, H) to $(\alpha, H - P(H))$. If we also recall the bound $\|P(H)\| \geq a\|H\|$ for all $H \in R_1$ from Lemma 1, this gives

$$\int_{R_1 \cap B_r} \frac{C}{\|H\|^{(k-1)(|\Psi|-|\Psi_1|)}} \frac{|\sum_{\sigma \in \mathcal{V}_1} \text{sgn}(\sigma)e^{i(\sigma(H-P(H)),H_0')}|^{2k}dH}{|\prod_{\alpha \in \Psi_1^+}(\alpha, H - P(H))|^{2k-2}}$$

$$\leq \int_{R_1 \cap B_r} \frac{C'}{\|P(H)\|^{(k-1)(|\Psi|-|\Psi_1|)}} \frac{|\sum_{\sigma \in \mathcal{V}_1} \text{sgn}(\sigma)e^{i(\sigma(H-P(H)),H_0')}|^{2k}dH}{|\prod_{\alpha \in \Psi_1^+}(\alpha, H - P(H))|^{2k-2}}.$$

P maps onto a one dimensional subspace, say $\text{span } v_1$, $\|v_1\| = 1$. We can do a change of variables so that we are integrating first with respect to $H' = H - P(H) \in \mathfrak{c}^+$ and then s, where $P(H) = sv_1$. If a and b are as in Lemma 1 then $s \geq a$ and $\|(I - P)H\| \leq b\|P(H)\|$ for $H \in R_1$. Note that $H \mapsto (P(H), (I - P)H)$ is an orthogonal change of variables so the Jacobian is a constant.

If we now use Fubini's Theorem to rewrite our integral (and forget the constant) we get

$$\int_a^r \frac{1}{s^{(k-1)(|\Psi|-|\Psi_1|)}} \int_{\mathfrak{c}^+ \cap B_{bs}} \frac{|\sum_{\sigma \in \mathcal{V}_1} \text{sgn}(\sigma)e^{i(\sigma(H'),H_0')}|^{2k}dH'}{|\prod_{\alpha \in \Psi_1^+}(\alpha, H')|^{2k-2}}ds.$$

Note that no element of Ψ annihilates $\sigma_j(H_0)$, so it is regular. It follows that no element of Φ_1 annihilates H_0'. Thus we can apply the induction hypothesis. Since $m < n$,

$$1 + \frac{m}{|\Psi|} = 1 + \frac{n}{|\Phi|} + (\frac{m}{|\Psi|} - \frac{n}{|\Phi|}) > 1 + \frac{n}{\Phi} + \epsilon_0.$$

So if $k < 1 + \frac{n}{|\Phi|} + \epsilon_0$ we have that the above integral is at most

$$\int_a^r \frac{1}{s^{(k-1)(|\Psi|-|\Psi_1|)}}O(s^{m-1-|\Psi_1|(k-1)})dr$$
$$= O(r^{m-|\Psi|(k-1)}).$$

At some point in our induction we get that $n = m$ and $\Psi = \Phi$. At this point our full induction hypothesis does not hold, but we have that

actual integral we are interested in is at most

$$\int_\delta^r O(r^{n-1-|\Phi|(k-1)})dr$$

if $k < 1 + \frac{n}{|\Phi|} + \epsilon_0$. This integral converges if

$$1 + \frac{n}{|\Phi|} + \epsilon_0 > k > 1 + \frac{n}{|\Phi|}.$$

Hence $\hat{\mu}_{H_0} \in L^2(\mathfrak{g})$ if $k > 1 + \frac{n}{|\Phi|}$.

Now we show the necessity of the condition $k > 1 + \frac{n}{|\Phi|}$.

We can rewrite

$$\frac{|\sum_{\sigma \in \mathcal{W}} \operatorname{sgn}(\sigma) e^{i(\sigma(H), H_0)}|^{2k}}{|\prod_{\alpha \in \Phi^+} (\alpha, H)|^{2k-2}}$$

as

$$\frac{1}{\|H\|^{|\Phi|(k-1)}} \frac{|\sum_{\sigma \in \mathcal{W}} \operatorname{sgn}(\sigma) e^{i\|H\|\left(\sigma(\frac{H}{\|H\|}), H_0\right)}|^{2k}}{\left|\prod_{\alpha \in \Phi^+} \left(\alpha, \frac{H}{\|H\|}\right)\right|^{2k-2}}$$

and consider this as $r^{-|\Phi|(k-1)} f(r, \phi_1, \ldots, \phi_{n-1})$, where f is a function in polar coordinates.

$$f(r, \phi_1, \ldots, \phi_{n-1}) = \frac{|\sum_{\sigma \in \mathcal{W}} \operatorname{sgn}(\sigma) e^{ir((1,\phi_1,\ldots,\phi_{n-1}),H_0)}|^{2k}}{|\prod_{\alpha \in \Phi^+} (\alpha, (1, \phi_1, \ldots, \phi_{n-1}))|^{2k-2}}$$

As before, we will integrate in $\mathfrak{t}^+$ with a ball around the origin removed, so we will always assume $r \geq 1$.

If we fix $\Phi = (\phi_1, \ldots, \phi_{n-1})$, we see that $f_\Phi(r) := f(r, \phi_1, \ldots, \phi_{n-1})$ is (the absolute value of) the sum of continuous functions that are periodic in r. Thus f is almost periodic in r.

Since $\|\mu_{H_0}^k\| \neq 0$ and f is continuous, we can find a point $(r_0, \psi_1, \ldots, \psi_{n-1})$, a $\delta > 0$ and a $\epsilon > 0$ so that if

$$U = \{(r, \phi_1, \ldots, \phi_{n-1}) : |\|\phi_i - \psi_i\| \leq \delta \forall i, |r - r_0| \leq \delta\} \subset \mathfrak{t}^+$$

then $f > 2\epsilon > 0$ on U.

We will have to change to polar coordinates to use this observation. The Jacobian of this change of variables is

$$\Delta = r^{n-1} \sin^{n-2} \phi_1 \ldots \sin^{n-2} \phi_{n-1}.$$

If necessary, we can modify U so that $|\Delta| \geq Cr^{n-1}$ on U, for some constant C. We then get that our integral greater or equal to

$$\int_{\psi_1-\delta}^{\psi_1+\delta} \cdots \int_{\psi_{n-1}-\delta}^{\psi_{n-1}+\delta} \int_1^\infty C\frac{1}{r^{|\Phi|(k-1)}} f(r, \phi_1, \ldots, \phi_{n-1}) r^{(n-1)} dr d\phi_{n-1} \ldots d\phi_1.$$

Say that f_Φ has an ϵ almost period in every interval of size M. Pick $N \geq M + 2\delta$. We know that $f_\Phi(r) \geq 2\epsilon$ on $[r_0 - \delta, r_0 + \delta]$. Pick τ_n, an ϵ almost period of f_Φ in the interval $[nN - r_0 + \delta, (n+1)N - r_0 - \delta]$, where $n > 0$. Hence $f_\Phi \geq \epsilon$ on $[r_0 + \tau_n - \delta, r_0 + \tau_n + \delta] \subset [nN, (n+1)N]$. If χ_F is the indicator function of $F = \bigcup_n [r_0 + \tau_n - \delta, r_0 + \tau_n + \delta]$, then the inner integral is at least

$$\int_1^\infty C\epsilon\chi_F r^{-(k-1)|\Phi|+n-1} dr \geq \int_1^\infty C\epsilon\chi_E r^{-(k-1)|\Phi|+n-1} dr$$

where $E = \bigcup_n [nN, nN + 2\delta]$. This integral is at least

$$\sum_{n=1}^\infty 2\delta C\epsilon(nN)^{-(k-1)|\Phi|+n-1}.$$

If $k \leq 1 + \frac{n}{|\Phi|}$, this diverges. Thus the inner integral is infinite for all $\phi_1, \ldots, \phi_{n-1}$ in the appropriate range. So if $k \leq 1 + \frac{n}{|\Phi|}$, our integral is infinite and $\mu^k \notin L^2(\mathfrak{g})$. $\qquad\square$

Remark (1). *A similar result holds when $\Phi = \Phi_1 \times \ldots \times \Phi_m$ is reducible. Say that the number of simple roots in Φ_i is r_i, and the fundamental Weyl chamber of Φ_i is $\mathfrak{t}_i^+$. In this case the integrand splits to give*

$$\int_{\mathfrak{t}_1^+} \int_{\mathfrak{t}_2^+} \cdots \int_{\mathfrak{t}_m^+} \frac{|A_{H_0}(t_1 + t_2 + \ldots + t_m)|^{2k}}{|\prod_{\alpha\in\Phi^+} (\alpha, t_1 + t_2 + \ldots + t_m)|^{2k-2}} dt_m \ldots dt_1.$$

This factors as

$$\int_{\mathfrak{t}_1^+} \frac{A_{H_0}^{\Phi_1}(t_1)}{|\prod_{\alpha\in\Phi_1^+}(\alpha, t_1)|} dt_1 \int_{\mathfrak{t}_2^+} \frac{A_{H_0}^{\Phi_2}(t_2)}{|\prod_{\alpha\in\Phi_3^+}(\alpha, t_2)|} dt_2 \cdots \int_{\mathfrak{t}_m^+} \frac{A_{H_0}^{\Phi_m}(t_m)}{|\prod_{\alpha\in\Phi_m^+}(\alpha, t_m)|} dt_m.$$

Since none of these factors can be zero, this is finite iff all the integrals converge. Hence $\mu_{H_0} \in L^2(\mathfrak{g})$ iff

$$k > \max\{1 + \frac{r_1}{|\Phi_1|}, 1 + \frac{r_2}{|\Phi_2|}, \ldots, 1 + \frac{r_m}{|\Phi_m|}\}.$$

Remark (2). *A measure μ is called L^p-improving if there is some $p < 2$ such that the operator $T_\mu : f \mapsto \mu * f$ is bounded from $L^p(\mathfrak{g})$ to $L^2(\mathfrak{g})$. Using sophisticated arguments Ricci and Travaglini [9] prove that for a regular, orbital measure μ, T_μ maps $L^p(\mathfrak{g})$ to $L^2(\mathfrak{g})$ if and only if $p \geq 1 + \operatorname{rank}(\mathfrak{g})/(2\dim(\mathfrak{g}) - \operatorname{rank}(\mathfrak{g})) = p(\mathfrak{g})$. The same reasoning as given in [4] Corollary 12 shows that our arguments give the weaker result: T_μ is bounded from $L^p(\mathfrak{g})$ to $L^2(\mathfrak{g})$ for any $p > p(\mathfrak{g})$.*

APPENDIX A

Φ	$\frac{n}{\|\Phi\|}$	Ψ	$\frac{m}{\|\Psi\|}$
A_n	$\frac{1}{n+1}$	$A_m, m < n$	$\frac{1}{m+1}$
B_n	$\frac{1}{2n}$	$B_m, m < n$	$\frac{1}{2m}$
C_n	$\frac{1}{2n}$	$C_m, m < n$	$\frac{1}{2m}$
D_n	$\frac{1}{2(n-1)}$	$D_m, m < n$	$\frac{1}{2(m-1)}$
E_6	$\frac{1}{12}$	$D_m, m < 6$	$\frac{1}{2(m-1)}$
E_7	$\frac{1}{18}$	$D_m, m < 7$	$\frac{1}{2(m-1)}$
E_7	$\frac{1}{18}$	E_6	$\frac{1}{12}$
E_8	$\frac{1}{30}$	$D_m, m < 8$	$\frac{1}{2(m-1)}$
E_8	$\frac{1}{30}$	E_6	$\frac{1}{12}$
E_8	$\frac{1}{30}$	E_7	$\frac{1}{18}$
F_4	$\frac{1}{12}$	$B_m, m < 4$	$\frac{1}{2m}$
G_2	$\frac{1}{6}$	A_1	$\frac{1}{2}$

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